ON RECONSTRUCTION OF LAMÉ COEFFICIENTS FROM PARTIAL CAUCHY DATA IN THREE DIMENSIONS

OLEG YU. IMANUVILOV, GUNTHER UHLMANN, AND MASAHIRO YAMAMOTO

Abstract. For the isotropic Lamé system, we prove in dimensions three or larger that both Lamé coefficients are uniquely recovered from partial Cauchy data on an arbitrary open subset of the boundary provided that the coefficient μ is a priori close to a constant.

In a bounded domain $\Omega \subset \mathbf{R}^N, N \ge 3$ with smooth boundary we consider the isotropic Lamé system:

$$\sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(C_{ijkl}\frac{\partial u_k}{\partial x_l}\right) = 0 \quad \text{in } \Omega,\ 1 \le i \le N \tag{0.1}$$

and

$$u|_{\partial\Omega} = f, \tag{0.2}$$

where

$$C_{ijkl} = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad 1 \le i,j,k,l \le N$$

where the Krönecker delta is denoted by δ_{ij}. The functions λ and μ are called the Lamé coefficients, $u(x) = (u_1(x), \dots, u_N(x))$ is the displacement. Assume that

$$\mu(x) > 0 \quad \text{on } \overline{\Omega}, \quad (\lambda + \mu)(x) > 0 \quad \text{on } \overline{\Omega}. \tag{0.3}$$

We set

$$\Lambda_{\lambda,\mu}u = \left(\sum_{j,k,l=1}^{N} \nu_j C_{1jkl}\frac{\partial u_k}{\partial x_l}, \dots, \sum_{j,k,l=1}^{N} \nu_j C_{Njkl}\frac{\partial u_k}{\partial x_l}\right),$$

where $\nu = (\nu_1, \dots, \nu_N)$ is the outward unit normal vector to $\partial\Omega$. The map $\Lambda_{\lambda,\mu}u$ is called the Dirichlet-to-Neumann map that maps the displacement at the boundary to the traction. Denote

$$\mathcal{L}_{\lambda,\mu}(x,D)u = \left(\sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(C_{1jkl}\frac{\partial u_k}{\partial x_l}\right), \dots, \sum_{j,k,l=1}^{N} \frac{\partial}{\partial x_j}\left(C_{Njkl}\frac{\partial u_k}{\partial x_l}\right)\right).$$

The partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ is defined by

$$\mathcal{C}_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}u)|_{\widetilde{\Gamma}};\ \mathcal{L}_{\lambda,\mu}(x,D)u = 0 \quad \text{in } \Omega,\ u|_{\partial\Omega} = f, \quad \text{supp}\, f \subset \widetilde{\Gamma},\ f \in H^{\frac{3}{2}}(\partial\Omega)\}.$$

Here $\widetilde{\Gamma}$ is an arbitrarily fixed open subset of $\partial\Omega$. We set $\Gamma_0 = \partial\Omega \setminus \overline{\widetilde{\Gamma}}$.

In this paper, we consider the following inverse problem: *Suppose that the partial Cauchy data $\mathcal{C}_{\lambda,\mu}$ are given. Can we determine the Lamé coefficients λ and μ?*

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This inverse problem has been studied since the 90's. A linearized version of this inverse problem for full data was studied in [6]. In two dimensions, Akamatsu, Nakamura and Steinberg [1] proved that for the case of full Cauchy data ($\widetilde{\Gamma} = \partial\Omega$) one can recover the Taylor series of the Lamé parameters on the boundary provided that the Lamé coefficients are C^∞ functions. This boundary determination result was extended in [13] to higher dimensions. In [11] Nakamura and Uhlmann, for the case of full Cauchy data, established that in the two dimensions the Lamé coefficients are uniquely determined, assuming that they are sufficiently close to a pair of positive constants. Recently Imanuvilov and Yamamoto in [9] proved for the two dimensional case that the Lamé coefficient λ can be recovered from partial Cauchy data if the coefficient μ is some positive constant. For the three dimensional case Nakamura and Uhlmann in [12], [14] and independently in [5] Eskin and Ralston proved uniqueness for both Lamé coefficients provided that μ is close to a positive constant. The proofs in the above papers rely on construction of complex geometric optics solutions. On the other hand, unlike the case of the Schrödinger operator, for partial Cauchy data, the construction of such a solutions for the isotropic Lamé system seems to be possible only for a dense set of Lamé coefficients.

The recovery of Lamé coefficients by partial Cauchy data on an arbitrary subboundary is useful from the practical point of view, because one can limit input and measurement subsets of $\partial\Omega$ as much as possible. To the best of our knowledge, there are no results on the unique recovery of the Lamé coefficients from the partial Cauchy data in the three dimensional case. The purpose of this article is to prove such uniqueness in three dimensions.

Finally we mention that this inverse problem is closely related to the method known as Electrical Impedance Tomography (EIT). EIT is used in prospection of oil and minerals and in medical imaging in detecting breast cancer, pulmonary edema, etc. For the mathematical treatments of this problem, we refer to [2], [3], [4], [7], [8], [10], [15] and the review paper [16].

Our main result is the following theorem.

Theorem 0.1. *Let μ_1, μ_2 be some positive constants and $\lambda_1, \lambda_2 \in C^\infty(\overline{\Omega})$ be some functions satisfying (0.3) and $\lambda_1 = \lambda_2$ on Γ_0. If $\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}$, then $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.*

Proof. The proof consists in showing that from partial Cauchy data one can recover the full Cauchy data. First following [13] we obtain that

$$(\lambda_1, \mu_1) = (\lambda_2, \mu_2) \quad \text{on } \widetilde{\Gamma}. \tag{0.4}$$

Let $u_j \in H^2(\Omega)$ be a functions such that

$$\mathcal{L}_{\lambda_j,\mu_j}(x, D)u_j = 0 \quad \text{in } \Omega, \quad u_j|_{\partial\Omega} = f, \tag{0.5}$$

where $\text{supp} f \subset \widetilde{\Gamma}$. Since the partial Cauchy data are the same, we obtain

$$\Lambda_{\lambda_1,\mu_1} u_1 = \Lambda_{\lambda_2,\mu_2} u_2 \quad \text{on } \widetilde{\Gamma}. \tag{0.6}$$

This equality combined with (0.4) implies that

$$(u_1, \frac{\partial u_1}{\partial \nu}) = (u_2, \frac{\partial u_2}{\partial \nu}) \quad \text{on } \widetilde{\Gamma}. \tag{0.7}$$

Since the functions μ_j are assumed to be constants, from (0.4) we conclude that

$$\mu_1 = \mu_2 \quad \text{on } \Omega. \tag{0.8}$$

For constant μ, we note that

$$\mathcal{L}_{\lambda,\mu}(x, D)u = \mu\Delta u + (\mu + \lambda)\nabla \mathrm{div}\, u + (\mathrm{div}\, u)\nabla\lambda.$$

Applying to equation (0.5) the operator rot and using the fact that μ_j is constant, we obtain

$$\mu_j \Delta \mathrm{rot} u_j = 0 \quad \text{in } \Omega. \tag{0.9}$$

From (0.4), (0.7) and equation (0.5) we conclude

$$(u_1, \frac{\partial u_1}{\partial \nu}, \partial_{x_i x_k} u_1) = (u_2, \frac{\partial u_2}{\partial \nu}, \partial_{x_i x_k} u_2) \quad \text{on } \widetilde{\Gamma}, \quad \forall i, k \in \{1, 2, 3\}. \tag{0.10}$$

Hence

$$(\mathrm{rot}\, u_1, \frac{\partial \mathrm{rot}\, u_1}{\partial \nu}) = (\mathrm{rot}\, u_2, \frac{\partial \mathrm{rot}\, u_2}{\partial \nu}) \quad \text{on } \widetilde{\Gamma}. \tag{0.11}$$

Equality (0.11) and the uniqueness of the solution for the Cauchy problem for the Laplace equation imply

$$\mathrm{rot}\, u_1 = \mathrm{rot}\, u_2 \quad \text{in } \Omega. \tag{0.12}$$

The Lamé operator, with the coefficient $\mu = const$, can be written in the form $\mathcal{L}(x, D)u = \nabla((\lambda + 2\mu)\mathrm{div}\, u) - \mu \mathrm{rot}\ \mathrm{rot} u$. Then using (0.8), (0.12) we obtain

$$\nabla((\lambda_1 + 2\mu_1)\mathrm{div}\, u_1) = \nabla((\lambda_2 + 2\mu_1)\mathrm{div}\, u_2) \quad \text{in } \Omega. \tag{0.13}$$

Since $(\lambda_1 + 2\mu_1)\mathrm{div}\, u_1 = (\lambda_2 + 2\mu_1)\mathrm{div}\, u_2$ on $\widetilde{\Gamma}$, equation (0.13) implies

$$(\lambda_1 + 2\mu_1)\mathrm{div}\, u_1 = (\lambda_2 + 2\mu_1)\mathrm{div}\, u_2 \quad \text{in } \Omega. \tag{0.14}$$

From (0.8), (0.14), (0.12) and the assumption $(\lambda_1 - \lambda_2)|_{\Gamma_0} = 0$ we conclude

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} \quad \text{on } \Gamma_0. \tag{0.15}$$

Therefore if $\mathrm{supp}\, f \subset \widetilde{\Gamma}$ in (0.4)and $f \in H^{\frac{3}{2}}(\partial\Omega)$ then $\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu}$ on $\partial\Omega$.

Next let $f \in H^{\frac{3}{2}}(\partial\Omega)$ and the functions $v_j \in H^2(\Omega)$ be solutions of the following boundary value problem

$$\mathcal{L}_{\lambda_j,\mu_j}(x, D)v_j = 0 \quad \text{in}\, \Omega, \quad v_j|_{\partial\Omega} = f, \quad j \in \{1, 2\}. \tag{0.16}$$

We claim that

$$\frac{\partial v_1}{\partial \nu} = \frac{\partial v_1}{\partial \nu} \quad \text{on } \widetilde{\Gamma}. \tag{0.17}$$

Indeed, let $w_j \in H^2(\Omega)$ be a solution to the Lamé system

$$\mathcal{L}_{\lambda_j,\mu_j}(x,D)w_j = 0 \quad \text{in } \Omega, \quad v_j|_{\partial\Omega} = g, \quad j \in \{1,2\}, \tag{0.18}$$

where $g \in H^{\frac{3}{2}}(\partial\Omega)$ and $\operatorname{supp} g \subset \widetilde{\Gamma}$ is an arbitrary function. Taking the scalar product of equation (0.16) with w_j and integrating by parts, we have

$$0 = \int_\Omega (\mathcal{L}_{\lambda_j,\mu_j}(x,D)v_j, w_j)dx = \int_\Omega (v_j, \mathcal{L}_{\lambda_j,\mu_j}(x,D)w_j)dx + \int_{\partial\Omega} ((\Lambda_{\lambda,\mu}v_j, w_j) - (\Lambda_{\lambda,\mu}w_j, v_j))d\sigma$$
$$= \int_{\partial\Omega} ((\Lambda_{\lambda_j,\mu_j}v_j, g) - (\Lambda_{\lambda_j,\mu_j}w_j, f))d\sigma = \int_{\widetilde{\Gamma}} (\Lambda_{\lambda_j,\mu_j}v_j, g)d\sigma - \int_{\partial\Omega} (\Lambda_{\lambda_j,\mu_j}w_j, f)d\sigma$$
$$= \int_{\widetilde{\Gamma}} (\Lambda_{\lambda_j,\mu_j}v_j, g)d\sigma - \int_{\partial\Omega} (\Lambda_{\lambda_1,\mu_1}w_1, f)d\sigma,$$

where $d\sigma$ denotes the surface measure.

This integral identity implies

$$\Lambda_{\lambda_1,\mu_1}v_1 = \Lambda_{\lambda_2,\mu_2}v_2 \quad \text{on } \widetilde{\Gamma}.$$

Repeating the arguments (0.9)-(0.15) we conclude

$$\frac{\partial v_1}{\partial\nu} = \frac{\partial v_2}{\partial\nu} \quad \text{on}\,\Gamma_0. \tag{0.19}$$

Hence, by (0.15), (0.19) the following full Cauchy data are equal:

$$\widetilde{\mathcal{C}}_{\lambda_1,\mu_1} = \widetilde{\mathcal{C}}_{\lambda_2,\mu_2}$$

where

$$\widetilde{\mathcal{C}}_{\lambda,\mu} = \{(u, \Lambda_{\lambda,\mu}u)|_{\partial\Omega};\, \mathcal{L}_{\lambda,\mu}(x,D)u = 0 \quad \text{in } \Omega,\ u|_{\partial\Omega} = f,\ f \in H^{\frac{3}{2}}(\partial\Omega)\}.$$

Applying the result of [5], [12] and [14], we obtain that $\lambda_1 = \lambda_2$. $\square$

This result immediately implies a local result for μ near constant.

Corollary 0.1. *Let B be a bounded set in $C^\infty(\overline{\Omega})$, $\lambda_1, \lambda_2 \in B, \lambda_1 = \lambda_2$ on Γ_0 and $\lambda_j(x) > C > 0, \mu_j(x) > C > 0$ on $\overline{\Omega}$. There exist positive $\epsilon(B) > 0$ and positive sufficiently large number N such that if $\sum_{k=1}^2 \|\nabla\mu_k\|_{C^N(\overline{\Omega})} \le \epsilon(B)$ and $\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}$, then $(\lambda_1,\mu_1) = (\lambda_2,\mu_2)$.*

Proof. Our proof by contradiction. Suppose that the statement of the corollary is false. Then there exists a sequence of positive $\{\epsilon_j\}_{j=1}^\infty$ such that $\epsilon_j \to 0$ and for each ϵ_j there exists $\{(\lambda_{k,j}, \mu_{k,j})\}$ such that

$$\{\lambda_{k,j}\}_{j=1}^\infty \subset B,\ k = \{1,2\}, \quad \text{and} \quad \sum_{k=1}^2 \|\nabla\mu_{k,j}\|_{C^N(\overline{\Omega})} \le \epsilon_j. \tag{0.20}$$

and

$$\mathcal{C}_{\lambda_{1,j},\mu_{1,j}} = \mathcal{C}_{\lambda_{2,j},\mu_{2,j}} \quad \forall j \in \{1,\dots,\infty\}. \tag{0.21}$$

By (0.20) and (0.21) there exist $\lambda_1, \lambda_2 \in C^\infty(\overline{\Omega})$ and positive constants μ_k such that

$$\mathcal{C}_{\lambda_1,\mu_1} = \mathcal{C}_{\lambda_2,\mu_2}.$$

Applying Theorem 0.1, we complete the proof of the corollary. $\square$

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Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins CO, 80523 USA, e-mail: oleg@math.colostate.edu

Department mathematics, UC Irvine, Irvine CA 92697, Department of Mathematics, University of Washington, Seattle, WA 98195 USA, e-mail: gunther@math.washington.edu

Department of Mathematical Sciences, The University of Tokyo, Komaba, Meguro, Tokyo 153, Japan, e-mail: myama@ms.u-tokyo.ac.jp